

Probability Theory, Ph.D Qualifying, Fall 2020

Completely solve any five problems.

1. Show that for any two random variables X and Y with $\text{Var}(X) < \infty$,

$$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}[E(X|Y)].$$

2. Suppose that X and Y are independent random variables with the same exponential density

$$f(x) = \theta e^{-\theta x}, \quad x > 0.$$

Show that the sum $X + Y$ and the ratio X/Y are independent.

3. Let X be a real valued random variable with mean value μ and variance σ^2 . Show that for $x > 0$,

$$P(X - \mu \geq x) \leq \frac{\sigma^2}{\sigma^2 + x^2}.$$

4. Let $X, \{X_n, n \geq 1\}, \{Y^{(k)}, k \geq 1\}, \{Y_n^{(k)}, n \geq 1, k \geq 1\}$, be real random variables such that $Y_n^{(k)} \rightarrow Y^{(k)}$ in distribution as $n \rightarrow \infty$, for fixed k , and that $Y^{(k)} \rightarrow X$ in distribution as $k \rightarrow \infty$. Show that $X_n \rightarrow X$ in distribution if

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} E[\min(|Y_n^{(k)} - X_n|, 1)] = 0.$$

5. Let $(X_n)_{n \geq 1}$ be a sequence of independent, real valued random variables with mean $\mu(X_n) = 0$. Let $S_n = X_1 + \dots + X_n$. Show that for all $a > 0$:

$$P\{\max_{k \leq n} |S_k| > a\} \leq \frac{\text{Var}(S_n)}{a^2}.$$

6. Let $X_1, X_2, \dots$ be a sequence of independent r.v.s with $EX_i = 0$. Let $S_n = X_1 + X_2 + \dots + X_n$ and $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$. Show that $\phi(S_n)$ is an $\mathcal{F}_n$ -submartingale for any convex ϕ provided that $E|\phi(S_n)| < \infty$ for all n .