

Ph.D. Prelim: Probability Theory, January 2003
(Solve at least 5 problems completely.)

1. (a) If $\{A_n, n \geq 1\}$ are events satisfying $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$ and $\lim_{n \rightarrow \infty} P(A_n) = 0$, then $P(A_n, i.o.) = 0$.

(b) If the independent random variables $X_1, \dots, X_n, \dots$ satisfy the condition

$$\text{Var}(X_i) \leq c < \infty, \quad i = 1, 2, \dots,$$

then the SLLN holds.

2. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteristic functions.

(b) Let $\{X_n\}$ be iid r.v.s with distribution $F(x)$ having finite mean μ and variance σ^2 . Let $S_n = X_1 + \dots + X_n$. Show that

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \rightarrow N(0, 1) \text{ in distribution as } n \rightarrow \infty.$$

3. Let $\{X_n, n \geq 1\}$ be a sequence of random variables such that

$$P(0 \leq X_{n+1} \leq X_n) = 1, \quad n \geq 1.$$

Prove that the convergence in probability $X_n \rightarrow 0$ as $n \rightarrow \infty$ implies

$$P(X_n \rightarrow 0 \text{ as } n \rightarrow \infty) = 1.$$

4. If F_n and F are distribution functions, $F_n \rightarrow F$ in distribution, and F continuous, show that

$$\sup_x |F_n(x) - F(x)| \rightarrow 0.$$

5. Let $\{X_n\}$ denote a sequence of iid random variables. Show that $E \sup_n |X_n/n| < \infty$ if and only if $E(|X_1| \cdot (\log |X_1|)^+) < \infty$.

6. Given iid random variables $\{X_n\}$, let

$$\overline{X} = \frac{X_1 + \dots + X_n}{n}.$$

Prove that for every $1 \leq k \leq n$, the equality

$$E[X_k | \overline{X}] = \overline{X}$$

holds with probability 1.