

Probability Theory, Ph.D Qualifying, Spring 2012

Completely solve any six problems.

1. If the independent random variables $\{X_n\}$ satisfy the condition

$$\text{Var}(X_i) \leq c < \infty, \quad i = 1, 2, \dots,$$

show that the SLLN holds.

2. (a) Let $(\Omega, \mathcal{F}, P)$ be a probability space. When do you say that $\mathcal{F}$ is P -trivial?
 (b) Show that a σ -field is P -trivial if and only if $\mathcal{F}$ is independent of itself. Also show in this case, that any $\mathcal{F}$ -random variable is constant almost surely.

3. Prove for iid random variables $\{X_n\}$ with $S_n = X_1 + \dots + X_n$ that

$$\frac{S_n - C_n}{n} \rightarrow 0 \text{ a.s.}$$

for some sequence of constants C_n if and only if $E|X_1| < \infty$.

4. Let $X_n, n \geq 1$, be a sequence of i.i.d nondegenerate real random variables, and put $S_n := X_1 + \dots + X_n, n \geq 1$. Show that:

- (a) $P(S_n \in B \text{ i.o.}) = 0$ or 1 , for any $B \in \mathcal{B}(\mathbb{R})$;
 (b) $\limsup_n S_n = \infty$ a.s or $-\infty$ a.s;
 (c) $\limsup_n (\pm S_n) = \infty$ a.s if the X_n are symmetric.

5. Let $X, \{X_n, n \geq 1\}, \{Y^{(k)}, k \geq 1\}, \{Y_n^{(k)}, n \geq 1, k \geq 1\}$, be real random variables such that $Y_n^{(k)} \rightarrow Y^{(k)}$ in distribution as $n \rightarrow \infty$, for fixed k , and that $Y^{(k)} \rightarrow X$ in distribution as $k \rightarrow \infty$. Show that $X_n \rightarrow X$ in distribution if $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} E[|Y_n^{(k)} - X_n| \wedge 1] = 0$.

6. (a) Define the uniform integrability of a family $X_t, t \in T$, of random variables.
 (b) Let $X \in L^1(\Omega, \mathcal{F}, P)$, and $\mathcal{A}_t, t \in T$ be a family of sub σ -fields of $\mathcal{F}$, where T is an index set. Show that the conditional expectations $\{E[X | \mathcal{A}_t], t \in T\}$ form a uniformly integrable family.

7. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with $E|X_1| < \infty$. Show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} E\left(\max_{1 \leq k \leq n} |X_k|\right) = 0.$$

8. Let $\{X_n\}$ be iid r.v.s with $E|X_1| < \infty$. Show that $\sum (-1)^n X_n/n$ converges a.s.
9. (a) Given a discrete-time filtration $(\Omega, \mathcal{F}, \mathcal{F}_n, n \geq 1)$, define a process X_n predictable w.r.t to the filtration.
 (b) Let $\mathcal{F}_n, n \geq 1$, be a filtration on $(\Omega, \mathcal{F}, P)$ and $X_n, \mathcal{F}_n, n \geq 1$, be any integrable adapted process.
 i. Show that X has an a.s unique decomposition $X_n = M_n + A_n, n \geq 1$, such that $\{M_n, \mathcal{F}_n\}$ is a martingale and $\{A_n, \mathcal{F}_n\}$ is a predictable process with $A_0 = 0$.
 ii. Show that X_n is a submartingale if and only if A_n is a.s nondecreasing (in n .)